

# PRECOMPACT GROUPS AND PROPERTY (T)

M. FERRER, S. HERNÁNDEZ, AND V. USPENSKIJ

ABSTRACT. For any topological group  $G$  the dual object  $\widehat{G}$  is defined as the set of equivalence classes of irreducible unitary representations of  $G$  equipped with the Fell topology. If  $G$  is compact,  $\widehat{G}$  is discrete, and we investigate to what extent this remains true for precompact groups, i.e. for dense subgroups of compact groups. We find that: (a) if  $G$  is a metrizable precompact group, then  $\widehat{G}$  is discrete; (b) if  $G$  is a countable non-metrizable precompact group, then  $\widehat{G}$  is not discrete; (c) every non-metrizable compact group contains a dense subgroup  $G$  for which  $\widehat{G}$  is not discrete. This generalizes to the non-Abelian case what was known for Abelian groups.

Kazhdan's property (T) can be defined in similar terms, but we must consider representations without non-zero invariant vectors rather than irreducible representations. If  $G$  is any countable Abelian precompact group, then  $G$  does not have property (T), although  $\widehat{G}$  is discrete if  $G$  is metrizable.

## 1. INTRODUCTION

For a topological group  $G$  let  $\widehat{G}$  be the set of equivalence classes of irreducible unitary representations of  $G$ . The set  $\widehat{G}$  can be equipped with a natural topology, the so-called Fell topology (see Section 2 for a definition).

A topological group  $H$  is *precompact* if it is isomorphic (as a topological group) to a subgroup of a compact group  $G$  (we may assume that  $H$  is dense in  $G$ ). If  $G$  is compact, then  $\widehat{G}$  is discrete. If  $H$  is a dense subgroup of  $G$ , the natural mapping  $\widehat{G} \rightarrow \widehat{H}$  is a bijection but in general need not be a homeomorphism. In the present paper we

---

*Date:* 4 December 2011.

The first and second listed authors acknowledge partial support by the Spanish Ministry of Science, grant MTM2008-04599/MTM; and by Fundació Caixa Castelló (Bancaixa), grant P1.1B2008-26.

*2010 Mathematics Subject Classification.* Primary 43A40. Secondary 22A25, 22C05, 22D35, 43A35, 43A65, 54H11

*Key Words and Phrases:* compact group, precompact group, representation, Pontryagin–van Kampen duality, compact-open topology, Fell dual space, Fell topology, Bohr compactification, Kazhdan property (T), determined group.

investigate the following question: if  $G$  is a compact group and  $H$  is a dense subgroup of  $G$ , under what conditions is the natural mapping  $\widehat{G} \rightarrow \widehat{H}$  a homeomorphism? Equivalently, for what precompact groups  $H$  is  $\widehat{H}$  discrete?

In the Abelian case the situation has been clarified in the work of several authors. If  $G$  is an Abelian topological group,  $\widehat{G}$  can be viewed as the group of all continuous homomorphisms  $G \rightarrow \mathbb{U}(1)$  equipped with the compact-open topology, where  $\mathbb{U}(1) = \{z \in \mathbb{C} : |z| = 1\}$ . Following [6], we say that a dense subgroup  $H$  of a topological Abelian group  $G$  *determines*  $G$  if the restriction homomorphism  $\widehat{G} \rightarrow \widehat{H}$  is a topological isomorphism. If every dense subgroup of  $G$  determines it, then  $G$  is called *determined*. A cornerstone for this topic is the following fact established by Aussenhofer [1] and, independently, Chasco [4]: every metrizable Abelian group  $G$  is determined. Moreover, if  $G$  is a compact metrizable Abelian group and  $H$  is dense in  $G$ , the proof of Außenhofer-Chasco theorem shows that there exists a compact subset  $K$  of  $H$  such that

$$K^{\triangleright} = \{\chi \in \widehat{G} : \forall x \in K \ |\chi(x) - 1| \leq \sqrt{2}\} = \{e\}.$$

Therefore, there is a single compact subset  $K$  in  $H$  that equips  $\widehat{G}$  with the discrete topology. The compact set  $K$  can be chosen as the set of points of a sequence converging to the unity  $e$ . This result is analogous to the classical Banach–Dieudonné theorem [3, IV.24]: If  $E$  is a metrizable locally convex space, the compact-open topology on its dual  $E'$  coincides with the topology of  $\mathfrak{N}$ -convergence, where  $\mathfrak{N}$  is the collection of all compact subsets of  $E$  each of which is the set of points of a sequence converging to 0.

Comfort, Raczkowski and Trigos-Arrieta [6] noted that Außenhofer-Chasco theorem fails for non-metrizable Abelian groups  $G$  even when  $G$  is compact. More precisely,

they proved that every non-metrizable compact Abelian group  $G$  of weight  $\geq \mathfrak{c}$  contains a dense subgroup that does not determine  $G$ . Hence, under the assumption of the continuum hypothesis, every determined compact Abelian group  $G$  is metrizable. Subsequently, it was shown in [13] that the result also holds without assuming the continuum hypothesis (see also [7]).

Our goal is to extend the results quoted above to compact groups that are not necessarily Abelian.

A certain extension of the Außenhofer-Chasco theorem to non-Abelian groups is due to Lukács [17]. He proved the following. Let  $G$  be a metrizable group,  $H$  a dense subgroup of  $G$ , and  $L$  a compact Lie group. Then the spaces of continuous homomorphisms  $CHom(G, L)$  and  $CHom(H, L)$ , equipped with the compact-open topology, are naturally homeomorphic.

Let  $H$  be a dense subgroup of a compact group  $G$ . We say that  $H$  *determines*  $G$  if  $\widehat{H}$  is discrete. A compact group  $G$  is *determined* if every dense subgroup of  $G$  determines  $G$ .

The Fell topology is defined on every set  $\mathcal{R}$  of equivalence classes of unitary representations (which may be reducible) of a given topological group  $G$ . Let  $1_G$  be the class of the trivial representation. The group  $G$  has *property (T)* if  $1_G$  is isolated in  $\mathcal{R} \cup \{1_G\}$  for every set  $\mathcal{R}$  of equivalence classes of unitary representations of  $G$  without non-zero invariant vectors. According to Proposition 1.2.3 in [2], this definition is equivalent to the definition of property (T) in terms of Kazhdan pairs which we remind below in Section 2. Compact groups have property (T), and we are interested in property (T) for precompact groups.

We now formulate our main results.

**Theorem 3.1.** *If  $H$  is a precompact metrizable group, then  $\widehat{H}$  is discrete.*

Equivalently, *every metrizable compact group is determined.* This extends to non-Abelian compact groups the result obtained by Aussenhofer and Chasco [1, 4] for Abelian metrizable groups.

**Theorem 4.1.** *If  $H$  is a countable precompact non-metrizable group, then  $1_H$  is not an isolated point in  $\widehat{H}$ .*

It follows that *every countable precompact group with property (T) is metrizable.* This implies Wang's result [22]: if a discrete group  $H$  has property (T), then its Bohr compactification  $bH$  is metrizable.

Theorem 4.1 also shows that *if  $H$  is a countable dense subgroup of a non-metrizable compact group  $G$ , then  $H$  does not determine  $G$ .*

**Theorem 4.2.** *If  $G$  is a non-metrizable compact group, then  $G$  has a dense subgroup  $H$  such that  $\widehat{H}$  is not discrete.*

Together with Theorem 3.1 this shows that *a compact group is determined if and only if it is metrizable.* This extends to non-Abelian compact groups the results given by Comfort, Raczkoski and Trigos-Arrieta [6], Hernández, Trigos-Arrieta and Macario [13], and Dikranjan, Shakhmatov [7] for Abelian compact groups.

**Theorem 5.1.** *If  $H$  is a countable Abelian precompact group, then  $H$  does not have property (T).*

We do not know whether there exists a non-compact precompact Abelian group with property (T) (Question 6.1). According to Theorem 5.1, such a group must be uncountable.

## 2. PRELIMINARIES: FELL TOPOLOGIES AND PROPERTY (T)

All topological groups are assumed to be Hausdorff. For a (complex) Hilbert space  $\mathcal{H}$  the unitary group  $U(\mathcal{H})$  of all linear isometries of  $\mathcal{H}$  is equipped with the strong operator topology (this is the topology of pointwise convergence). Then  $U(\mathcal{H})$  is a topological group.

A *unitary representation*  $\rho$  of the topological group  $G$  is a continuous homomorphism  $G \rightarrow U(\mathcal{H})$ , where  $\mathcal{H}$  is a complex Hilbert space. A closed linear subspace  $E \subseteq \mathcal{H}$  is an *invariant* subspace for  $\mathcal{S} \subseteq U(\mathcal{H})$  if  $ME \subseteq E$  for all  $M \in \mathcal{S}$ . If there is a closed subspace  $E$  with  $\{0\} \subsetneq E \subsetneq \mathcal{H}$  which is invariant for  $\mathcal{S}$ , then  $\mathcal{S}$  is called *reducible*; otherwise  $\mathcal{S}$  is *irreducible*. An *irreducible representation* of  $G$  is a unitary representation  $\rho$  such that  $\rho(G)$  is irreducible.

If  $\mathcal{H} = \mathbb{C}^n$ , we identify  $U(\mathcal{H})$  with the *unitary group of order  $n$* , that is, the compact Lie group of all complex  $n \times n$  matrices  $M$  for which  $M^{-1} = M^*$ . We denote this group by  $\mathbb{U}(n)$ .

Two unitary representations  $\rho : G \rightarrow U(\mathcal{H}_1)$  and  $\psi : G \rightarrow U(\mathcal{H}_2)$  are *equivalent* if there exists a Hilbert space isomorphism  $M : \mathcal{H}_1 \rightarrow \mathcal{H}_2$  such that  $\rho(x) = M^{-1}\psi(x)M$  for all  $x \in G$ . The *dual object* of a topological group  $G$  is the set  $\widehat{G}$  of equivalence classes of irreducible unitary representations of  $G$ .

If  $G$  is a precompact group, the Peter-Weyl Theorem (see [14]) implies that all irreducible unitary representation of  $G$  are finite-dimensional and determine an embedding of  $G$  into the product of unitary groups  $\mathbb{U}(n)$ .

The *Bohr topology* on a topological group  $G$  is the finest precompact group topology that is coarser than the given topology on  $G$ . The *Bohr compactification* of  $G$  is the completion of  $G$  equipped with the Bohr topology.

If  $\rho : G \rightarrow U(\mathcal{H})$  is a unitary representation, a complex-valued function  $f$  on  $G$  is called a *function of positive type* (or *positive-definite function*) associated with  $\rho$  if there exists a vector  $v \in \mathcal{H}$  such that  $f(g) = (\rho(g)v, v)$  (here  $(\cdot, \cdot)$  denotes the inner product in  $\mathcal{H}$ ). We denote by  $P'_\rho$  be the set of all functions of positive type associated with  $\rho$ . Let  $P_\rho$  be the convex cone generated by  $P'_\rho$ , that is, the set of sums of elements of  $P'_\rho$ .

Let  $G$  be a topological group,  $\mathcal{R}$  a set of equivalence classes of unitary representations of  $G$ . The *Fell topology* on  $\mathcal{R}$  is defined as follows: a typical neighborhood of  $[\rho] \in \mathcal{R}$  has the form

$$W(f_1, \dots, f_n, C, \epsilon) = \{[\sigma] \in \mathcal{R} : \exists g_1, \dots, g_n \in P_\sigma \forall x \in C |f_i(x) - g_i(x)| < \epsilon\},$$

where  $f_1, \dots, f_n \in P_\rho$  (or  $\in P'_\rho$ ),  $C$  is a compact subspace of  $G$ , and  $\epsilon > 0$ . In particular, the Fell topology is defined on the dual object  $\widehat{G}$ . If  $G$  is locally compact, the Fell topology on  $\widehat{G}$  can be derived from the Jacobson topology on the primitive ideal space of  $C^*(G)$ , the  $C^*$ -algebra of  $G$  [8, section 18], [2, Remark F.4.5].

Let  $\pi$  be a unitary representation of a topological group  $G$  on a Hilbert space  $\mathcal{H}$ . Let  $F \subseteq G$  and  $\epsilon > 0$ . A unit vector  $v \in \mathcal{H}$  is called  $(F, \epsilon)$ -invariant if  $\|\pi(g)v - v\| < \epsilon$  for every  $g \in F$ .

A topological group  $G$  has *property (T)* if there exist a pair  $(Q, \epsilon)$  (called a *Kazhdan pair*), where  $Q$  is a compact subset of  $G$  and  $\epsilon > 0$ , such that for every unitary representation  $\rho$  having a unit  $(F, \epsilon)$ -invariant vector there exists a non-zero invariant vector. This definition is equivalent to the one given in the previous section [2, Proposition 1.2.3].

We refer to Fell's papers [9, 10], the classical text by Dixmier [8] and the recent monographs by de la Harpe and Valette [12], and Bekka, de la Harpe and Valette [2] for basic definitions and results concerning Fell topologies and property (T).

### 3. PRECOMPACT METRIZABLE GROUPS

The aim of this section is to prove the following

**Theorem 3.1.** *If  $G$  is a precompact metrizable group, then  $\widehat{G}$  is discrete.*

We view  $G$  as a dense subgroup of a compact metrizable group  $K$ . Integrals over  $K$  will be taken with respect to the normalized Haar measure on  $K$ . The Hilbert space  $L^2(K)$  is constructed with the aid of the same measure. Up to equivalence, there are countably many irreducible unitary representations of  $K$ . Enumerate them as  $\rho_i$ ,  $i \in \mathbb{N}$ , let  $\chi_i$  be their characters,  $d_i$  their dimensions. Let  $P'_i = P'_{\rho_i}$  be the corresponding set of functions of positive type, let  $P_i$  be the convex cone generated by  $P'_i$ , and let  $Q_i \subset C(K)$  be the linear space generated by  $P_i$ . The spaces  $Q_i$  are finite-dimensional ( $\dim Q_i = d_i^2$ ) and pairwise orthogonal in the Hilbert space  $L^2(K)$ . Let  $N_i = \{f \in P_i : f(e) = 1\}$  be the space of normalized functions in  $P_i$ . This is a compact subset of the cone  $P_i$ . Let  $h_i = \chi_i/d_i \in N_i$  be the normalized character.

Recall that a function on  $G$  is *central* if it is constant on conjugacy classes.

**Lemma 3.2.** *The only central functions in  $Q_i$  are the functions  $c\chi_i$ ,  $c \in \mathbb{C}$ .*

*Proof.* Any central function  $g \in Q_i$  is the sum in  $L^2(K)$  of the series  $\sum c_j \chi_j$ , where  $c_j = \int_K g \bar{\chi}_j$ . Since  $Q_i \perp Q_j$ , we have  $c_j = 0$  for all  $j \neq i$ .  $\square$

**Lemma 3.3.** *Let  $X$  be compact,  $D$  a dense subset of  $X$ ,  $F$  a compact subset of  $C(X)$ . If  $g \in C(X)$  is at the distance  $> \epsilon$  from  $F$ , there exists a finite subset  $Y \subseteq D$  such that the distance from  $g|_Y$  to  $F|_Y$  in  $C(Y)$  is  $> \epsilon$ .*  $\square$

**Lemma 3.4.** *Let  $V$  be a measurable subset of  $K$ . Then  $\int_V \chi_i \rightarrow 0$  as  $i \rightarrow \infty$ .*

*Proof.* The integrals  $\int_V \chi_i$  are the scalar products of the characteristic function of  $V$  with the terms of the orthonormal sequence  $(\bar{\chi}_i)$  in  $L^2(K)$ .  $\square$

**Lemma 3.5.** *Let  $V \subseteq K$  be a compact neighborhood of the identity  $e$  that is invariant under inner automorphisms of  $K$ . Let  $f \in C(K)$  be a continuous central function. Let  $h_i = \chi_i/d_i \in N_i$ . Then the distance from  $f|_V$  to  $h_i|_V$  in the space  $C(V)$  is the same as the distance from  $f|_V$  to the compact set  $N_i|_V$ .*

*Proof.* Let  $h \in N_i$  be such that  $h|_V$  is as close to  $f|_V$  as possible. Averaging over  $K$  (that is, replacing  $h$  by  $h'$  defined by  $h'(x) = \int_{y \in K} h(yxy^{-1})$ ) we get a central function in  $N_i$  at the same distance from  $f|_V$ . According to Lemma 3.2, the only central function in  $N_i$  is  $h_i = \chi_i/d_i$ .  $\square$

We now are ready to prove Theorem 3.1.

*Proof.* Recall that we view  $G$  as a dense subgroup of a compact metrizable group  $K$ . Pick a bi-invariant (= invariant under left and right translations, hence also under inner automorphisms) metric  $b$  on  $K$ . We denote by  $V_\epsilon$  the closed  $\epsilon$ -ball (with respect to  $b$ ) centered at the neutral element  $e$ . We use the notation introduced above.

We can identify the sets  $\widehat{G}$  and  $\widehat{K}$  (not taking the topology of these sets into account). Let  $\rho$  be an irreducible unitary representation of  $K$ . We must show that  $[\rho]$  is an isolated point in  $\widehat{G}$ .



Let  $\chi = \chi_\rho$  be the character of  $\rho$ , and  $d = d_\rho$  its dimension. Consider the normalized character  $h = \chi/d$ . Pick  $\epsilon > 0$  such that  $\Re h(x) > 2/3$  for every  $x \in V_\epsilon$ . Let  $V = V_\epsilon$ . Since  $\int_V \chi_i \rightarrow 0$  (Lemma 3.4), for all sufficiently large indices  $i$  there exists a point  $x_i \in V$  such that  $\Re \chi_i(x) \leq 1/3$ . Choose  $x_i$  as close to  $e$  as possible. Applying the same argument for any smaller value of  $\epsilon$ , we see that  $x_i \rightarrow e$ .

Let  $h_i = \chi_i/d_i$  and  $\epsilon_i = b(x_i, e)$ . We have  $\epsilon_i \rightarrow 0$ . Pick  $\epsilon'_i > \epsilon_i$  so that  $\epsilon'_i \rightarrow 0$ . Denote  $V_i = V_{\epsilon_i}$ ,  $V'_i = V_{\epsilon'_i}$ . Then  $|h(x_i) - h_i(x_i)| > 1/3$  and hence  $\text{dist}(h|_{V_i}, h_i|_{V_i}) > 1/3$ . It follows from Lemma 3.5 that  $\text{dist}(h|_{V_i}, N_i|_{V_i}) > 1/3$ . We claim that there exists a finite set  $F_i \subseteq V'_i \cap G$  such that  $\text{dist}(h|_{F_i}, N_i|_{F_i}) > 1/3$ . If  $V_i$  is the closure of its interior, then  $V_i \cap G$  is dense in  $V_i$ , and it follows from Lemma 3.3 that we can find  $F_i \subseteq V_i \cap G$  with the required property. In general we apply Lemma 3.3 to the closure of the interior of  $V'_i$  in place of  $X$ .

Since  $\epsilon'_i \rightarrow 0$ , the union  $F$  of all the  $F_i$ 's plus the point  $e$  is a compact subset of  $G$ . If  $f_i \in P_i$  is such that the sequence  $(f_i|_F)$  uniformly converges to  $h|_F$ , then  $f_i(e) \rightarrow 1$ , and normalizing each  $f_i$  (that is, replacing each  $f_i$  by  $f'_i = f_i/f_i(e)$ ) we get a normalized sequence  $(f'_i)$  such that  $f'_i \in N_i$  and  $(f'_i|_F)$  uniformly converges to  $h|_F$ . This is impossible, since  $\text{dist}(h|_F, N_i|_F) \geq \text{dist}(h|_{F_i}, N_i|_{F_i}) > 1/3$ . We have proved that the neighborhood  $W(h|_G, F, 1/3)$  of  $[\rho]$  contains only finitely many elements of  $\widehat{G}$ . Since  $\widehat{G}$  is a  $T_1$ -space (this follows, for example, from Lemma 3.3), it follows that  $\widehat{G}$  is discrete.  $\square$

The compact set  $F = F_\rho \subseteq G$  that we constructed in the proof above is the set of points of a sequence converging to  $e$ . It depends on the point  $[\rho] \in \widehat{G}$  the neighborhood of which we are constructing. We observe that there exists a single compact subset  $Q \subseteq G$  such that for every  $[\rho_i] \in \widehat{G}$  the neighborhood  $W(h_i|_G, Q, 1/3)$  of  $[\rho_i]$  in  $\widehat{G}$  is

finite. It suffices to delete a finite set  $D_i$  from each  $F_{\rho_i}$  in such a way that the union  $Q = \bigcup (F_{\rho_i} \setminus D_i)$  is the set of points of a sequence converging to  $e$ .

#### 4. COUNTABLE PRECOMPACT GROUPS

In this section we prove the following:

**Theorem 4.1.** *If  $H$  is a countable precompact non-metrizable group, then  $1_H$  is not an isolated point in  $\widehat{H}$ .*

**Theorem 4.2.** *If  $G$  is a non-metrizable compact group, then  $G$  has a dense subgroup  $H$  such that  $\widehat{H}$  is not discrete.*

For a topological group  $G$  let  $\widehat{G}_n \subseteq \widehat{G}$  be the set of classes of  $n$ -dimensional irreducible unitary representations. We denote by  $w(X)$  the weight (= the least cardinality of a base) of a topological space  $X$ .

**Proposition 4.3.** *Let  $G$  be a topological group. Suppose that there exists an integer  $n$  such that  $w(K) < |\widehat{G}_n|$  for every compact subset  $K$  of  $G$ . Then there exists an integer  $m$  such that  $1_G$  is not an isolated point in  $\widehat{G}_m \cup \{1_G\}$ .*

*Proof.* Let  $\mathcal{F}_n$  be the set of classes of all  $n$ -dimensional unitary representations of  $G$  (which may be reducible). Let  $\mathcal{R}_n \subseteq \mathcal{F}_n$  be the set of classes of all  $n$ -dimensional representations without non-zero invariant vectors. Let  $k = n^2$ . We first prove that  $1_G$  is not isolated in  $\mathcal{R}_k \cup \{1_G\}$ .

Let  $F$  be a compact subset of  $G$  and  $\epsilon > 0$ . It suffices to prove that there exists a  $k$ -dimensional representation  $\tau$  of  $G$  without non-zero invariant vectors and a function  $f \in P'_\tau$  such that  $|f(x) - 1| < \epsilon$  for every  $x \in F$ . Equip  $\mathbb{U}(n)$  with any compatible bi-invariant metric, and equip  $C(F, \mathbb{U}(n))$  with the sup-metric. Since

$w(C(F, \mathbb{U}(n))) = w(F) < |\widehat{G}_n|$ , for every  $\delta > 0$  we can find two homomorphisms  $\rho_1, \rho_2 : G \rightarrow \mathbb{U}(n)$  which are  $\delta$ -close on  $F$  and determine non-equivalent irreducible representations of  $G$  on  $\mathbb{C}^n$  (see [11]). Indeed, the compact group  $\mathbb{U}(n)$  isometrically acts on  $C(F, \mathbb{U}(n))$  by conjugation, and the orbit space can be identified as a set with  $\mathcal{F}_n$ . Equip  $\mathcal{F}_n$  with the quotient metric (which need not be compatible with the Fell topology). Then the weight of the resulting metric space is  $\leq w(F) < |\widehat{G}_n|$ , hence the subset  $\widehat{G}_n$  cannot be discrete and contains a pair of distinct  $\delta$ -close points  $[\rho_1], [\rho_2]$ . The distance between  $[\rho_1]$  and  $[\rho_2]$  in  $\widehat{G}_n$  is equal to

$$\inf\{\text{dist}(\rho_1, A\rho_2A^{-1}) : A \in \mathbb{U}(n)\},$$

so replacing if necessary  $\rho_2$  by an equivalent representation, we may assume that  $\rho_1$  and  $\rho_2$  are  $\delta$ -close, as claimed.

Let  $E = \text{End } \mathbb{C}^n$  be the  $n^2$ -dimensional Hilbert space of endomorphisms of  $\mathbb{C}^n$ . The scalar product on  $E$  is given by the formula  $(A, B) = \text{Tr}(AB^*)/n$ . The formula  $\tau(g)A = \rho_1(g)A\rho_2^{-1}(g)$  ( $g \in G$ ,  $A \in E$ ) defines a unitary representation of  $K$  on  $E$  that does not contain non-zero invariant vectors. Indeed, if  $\tau(g)A = A$  for all  $A \in E$ , then  $A$  intertwines  $\rho_1$  and  $\rho_2$  and hence is zero in virtue of Shur's Lemma

Let

$$f(g) = \frac{1}{n} \text{Tr}(\rho_1(g)\rho_2^{-1}(g)) = (\tau(g)I, I),$$

where  $I \in E$  denotes the identity mapping on  $\mathbb{C}^n$ . Then  $f \in P_\tau$  is a function of positive type associated with  $\tau$ . If  $\delta > 0$  is small, then  $\rho_1(g)\rho_2^{-1}(g)$  is in a small neighborhood of 1 in  $\mathbb{U}(n)$  and hence  $|f(g) - 1| < \epsilon$  for every  $g \in F$ .

We have just shown that there exists a  $k$ -dimensional unitary representation of  $G$  without non-zero invariant vectors but containing an  $(F, \epsilon)$ -invariant unit vector  $v$ . Writing the representation as a direct sum of  $s$  irreducible representations, we can write

$v = v_1 + \cdots + v_s$ , where  $s \leq k$ , the vectors  $v_1, \dots, v_s$  are pairwise orthogonal, and each of them is moved by every  $g \in F$  by less than  $\epsilon$ . Since  $\sum \|v_i\|^2 = 1$ , one of the vectors  $v_i$  has length  $\geq 1/\sqrt{k} = 1/n$ . Suppose it is the vector  $v_1$ . Then  $v_1/\|v_1\|$  is a unit vector which is  $(F, n\epsilon)$  invariant. This shows that  $1_G$  is not isolated in  $\bigcup_{2 \leq m \leq k} \widehat{G}_m \cup \{1_G\}$  and hence also in some  $\widehat{G}_m \cup \{1_G\}$ .

□

Our proof yields the following result of Wang [22]:

**Corollary 4.4.** *Let  $G$  be a discrete group such that  $1_G$  is an isolated point in  $\bigcup_{n \in \mathbb{N}} \widehat{G}_n$ . Then  $\widehat{G}_n$  is finite for all  $n \in \mathbb{N}$ .*

Proposition 4.3 also implies the following

**Theorem 4.5.** *Let  $G$  be topological group,  $\kappa$  a cardinal such that  $w(K) \leq \kappa$  for every compact subset of  $G$ . If  $1_G$  is an isolated point in  $\widehat{G}_n \cup \{1_G\}$  for every  $n$ , then  $w(bG) \leq \kappa$ , where  $bG$  is the Bohr compactification of  $G$ .*

*Proof.* According to Proposition 4.3,  $|\widehat{G}_n| \leq \kappa$  for every  $n$ . The compact group  $bG$  therefore has  $\leq \kappa$  non-equivalent irreducible unitary representations and hence has weight  $\leq \kappa$  in virtue of the Peter–Weyl theorem. □

The case  $\kappa = \omega$  of the previous theorem deserves to be explicitly stated:

**Corollary 4.6.** *Let  $G$  be topological group such that every compact subset of  $G$  is metrizable. If  $1_G$  is an isolated point in  $\widehat{G}_n \cup \{1_G\}$  for every  $n$ , then the Bohr compactification of  $G$  is metrizable.*

Note that this Corollary implies the following assertion which is stronger than Theorem 4.1: *if  $G$  is a non-metrizable precompact group such that all compact subsets of  $G$  are metrizable, then  $1_G$  is not an isolated point in  $\widehat{G}$ .*

Our proof of Theorem 4.2 is based on the following

**Lemma 4.7.** *Every compact group of weight  $\omega_1$  has a dense countable subgroup.*

*Proof.* A proof can be found e.g. in [5]. For the reader's convenience we remind the proof. A compact space  $X$  is *dyadic* if there exists a mapping  $2^\kappa \rightarrow X$  of a Cantor cube  $2^\kappa$  onto  $X$ . All compact groups are dyadic (moreover, any compact  $G_\delta$ -subset of any topological group is dyadic, see [20, 21]), and any dyadic compact space of weight  $\leq \mathfrak{c} = 2^\omega$  is separable, being an image of the separable space  $2^\mathfrak{c}$ .  $\square$

We now prove the following stronger version of Theorem 4.2: *If  $G$  is a non-metrizable compact group, then  $G$  has a dense subgroup  $H$  such that  $1_H$  is not isolated in  $\widehat{H}$ .*

Embedding  $G$  into the product of unitary groups, we see that there exists a continuous homomorphism  $f : G \rightarrow G'$  onto a compact group of weight  $\omega_1$ . Let  $H'$  be a dense countable subgroup of  $G'$  (Lemma 4.7), and let  $H = f^{-1}(H')$ . Since  $f$  is open,  $H$  is dense in  $G$ . According to Theorem 4.1,  $1_{H'}$  is not isolated in  $\widehat{H'}$ . Since  $f^* : \widehat{H'} \rightarrow \widehat{H}$  is continuous, sends  $1_{H'}$  to  $1_H$  and  $\widehat{H'} \setminus \{1_{H'}\}$  to  $\widehat{H} \setminus \{1_H\}$ , it follows that  $1_H$  is not isolated in  $\widehat{H}$ .

## 5. THE PROPERTY (T)

The Kazhdan property (T) for topological groups was introduced in [15]. This property has several consequences on the structure of the groups that satisfy it. For  $\sigma$ -compact locally compact groups Property (T) is equivalent to the fixed point property

for isometric affine actions on real Hilbert spaces [2, Theorem 2.12.4]. See [2, 8, 12] for further details on this important topic.

We saw that for every metrizable precompact group  $G$  the dual  $\widehat{G}$  is discrete (Theorem 3.1). In contrast, we have the following result.

**Theorem 5.1.** *If  $G$  is an Abelian, countable precompact group, then  $G$  does not have property (T).*

The result is no longer true if “Abelian” is dropped. Indeed, certain compact Lie groups admit dense countable subgroups which have property (T) as discrete groups [2, Theorem 6.4.4] and hence also as precompact topological groups.

We begin with the following characterization of property (T) for precompact groups.

**Theorem 5.2.** *Let  $H$  be a dense subgroup of a compact group  $G$ . Then  $H$  has property (T) if and only if the following condition holds: there exist a compact subset  $K \subseteq H$  and  $\epsilon > 0$  such for every function  $f$  of positive type with zero integral over  $G$  (with respect to the Haar measure) the distance from  $f|_K$  to 1 in the space  $C(K)$  (with respect to the sup-norm) is  $\geq \epsilon$ .*

*Proof.* Functions of positive type on  $G$  are the functions of the form  $g \mapsto (\rho(g)v, v)$  associated with unitary representations  $\rho$  of  $G$  (here  $v$  is a vector in the Hilbert space  $\mathcal{H}$  of the representation). If a representation  $\rho$  does not contain the trivial representation (that is, there are no non-zero invariant vectors), the integral of  $(\rho(g)v, v)$  over  $G$  is zero, for every  $v \in \mathcal{H}$  (this follows, for example, from orthogonality relations). Conversely, if the integral of  $f(g) = (\rho(g)v, v)$  is zero, then  $v$  lies in  $\mathcal{H}_1 = \mathcal{H}_2^\perp$ , where  $\mathcal{H}_2 \subseteq \mathcal{H}$  is the subspace of invariant vectors. Therefore  $f$  is associated with a representation on

$V_1$  without non-zero invariant vectors. The theorem now follows from the definition of property (T), taking into account that the continuous unitary representations of  $G$  and  $H$  are in a one-to-one correspondence.  $\square$

**Corollary 5.3.** *Let  $G$  be a dense subgroup of a compact group  $H$ . If  $G$  has property (T), then there exist a compact subset  $K \subseteq G$ , a real (signed) measure  $\mu$  on  $K$  and a real number  $c$  such that for every real function  $f$  of positive type on  $H$  with  $f(e) = 1$  and zero integral over  $H$  we have*

$$\int_K f \mu < c < \int_K 1 \mu.$$

*Proof.* Consider the convex set  $P$  of all real functions  $f$  of positive type on  $H$  such that  $\int_H f = 0$  and  $f(e) = 1$ . Let  $K \subseteq G$  be a compact subset with the property described in Theorem 5.2. According to the theorem, the image of  $P$  under the restriction map  $f \mapsto f|_K$  is at a positive distance from 1. The Hahn-Banach theorem implies that 1 can be separated from the convex set  $P|_K$  by a linear functional.  $\square$

Following Lubotzky and Zimmer [16], if  $\mathcal{R}$  is a set of classes of representations of  $G$ , we say that  $G$  has *Property (T) with respect to  $\mathcal{R}$*  if  $1_G$  is isolated in  $\mathcal{R} \cup \{1_G\}$ .

*Remark 5.4.* Let  $G$  be a precompact group, and let  $G_d$  denote the same group with the discrete topology. Let  $\mathcal{R}$  be the set of equivalence classes of finite-dimensional unitary representations of  $G$  without non-zero invariant vectors. It is readily seen that the proof of Theorem 5.2 shows that  $G_d$  has property (T) with respect to  $\mathcal{R}$  if and only if there exist a finite subset  $K \subseteq G$  and  $\epsilon > 0$  such that for every function  $f$  of positive type with zero integral over the completion of  $G$  the distance from  $f|_K$  to 1 in the space  $C(K)$  (with respect to the sup-norm) is  $\geq \epsilon$ .

**Theorem 5.5.** *Let  $G$  be a precompact group such that every compact subset of  $G$  is countable, and let  $G_d$  be the same group equipped with the discrete topology. Let  $\mathcal{R}$  be the set of equivalence classes of finite-dimensional unitary representations of  $G$  without non-zero invariant vectors. The following assertions are equivalent:*

- (a)  $G$  has property (T);
- (b)  $G_d$  has property (T) with respect to  $\mathcal{R}$ .

*Proof.* (b)  $\Rightarrow$  (a) is clear.

(a)  $\Rightarrow$  (b). Let  $H$  be the completion of  $G$ . Suppose that  $G$  has property (T) but  $G_d$  does not have property (T) with respect to  $\mathcal{R}$ . Let  $P$  be the convex set that was used in the proof of Corollary 5.3:  $P$  is the set of all real-valued functions  $f$  of positive type on  $H$  with zero integral and such that  $f(e) = 1$ . Using Corollary 5.3, find a compact set  $K \subseteq G$  such that  $1|_K$  is not in the weak closure of the convex set  $P|_K$  in the Banach space  $C(K)$  (the weak closure and the norm closure are the same because of convexity, see [19, Theorem 3.12]). Since all compact subsets of  $G$  are countable, we can write  $K$  as the set of points of a sequence  $(x_n)$ . According to Remark 5.4, for every finite subset  $F$  of  $G$  and  $\epsilon > 0$  there exists  $f \in P$  such that the distance from  $f|_F$  to  $1|_F$  in the space  $C(F)$  is  $< \epsilon$ . Applying this to initial segments of the sequence  $(x_n)$ , we obtain a sequence  $(f_n)$  of functions  $\in P$  that converges pointwise to 1 on  $K$ . All functions in  $P$  are uniformly bounded by 1, so Lebesgue's dominated convergence theorem implies that the sequence  $(f_n)$  weakly converges to 1 in  $C(K)$ . Therefore,  $1|_K$  is in the weak closure of  $P|_K$ , contrary to our assumption.  $\square$

We are now in a position to prove Theorem 5.1:

*If  $G$  is an Abelian, countable precompact group, then  $G$  does not have property (T).*



*Proof.* Let  $H$  be the completion of  $G$ . In virtue of Theorem 5.5, we must verify that  $G_d$  does not have property (T) with respect to  $\mathcal{R}$ , where  $\mathcal{R}$  is the same as in Theorem 5.5. It suffices to prove that for every finite set  $F \subseteq G$  and  $\epsilon > 0$  there is a continuous character  $\chi : H \rightarrow \mathbb{U}(1)$ ,  $\chi \neq 1$ , such that  $|\chi(x) - 1| < \epsilon$  for every  $x \in F$ .

Assume the contrary. Let  $F \subseteq G$  be a finite set such that for some  $\epsilon > 0$  and every  $\chi \in \widehat{H} \setminus \{1\}$  the restriction  $\chi|_F$  is at the distance  $\geq \epsilon$  from  $1|_F$ . Then the restriction homomorphism  $\chi \mapsto \chi|_F$  from  $\widehat{H}$  to  $\mathbb{U}(1)^F$  is one-to-one and must be a homeomorphism onto its image, since our assumption means that its inverse is continuous at 1. This is a contradiction, since the compact group  $\mathbb{U}(1)^F$  cannot have infinite discrete subgroups.  $\square$

## 6. QUESTIONS

**Question 6.1.** *Does there exist a non-compact precompact Abelian group with property (T)?*

In virtue of Theorem 5.1 such a group must be uncountable.

The definition of the Fell topology, given in [2] and used in the present paper, is not the same as the definition given in [18]. Using the notation of Section 2, the difference is the following. In [18], functions  $f_1, \dots, f_n \in P'_\rho$  of positive type are approximated by functions  $g_1, \dots, g_n \in P'_\sigma$  of positive type rather than by their sums  $\in P_\rho$  that we allowed. For locally compact groups  $G$  the two definitions of the Fell topology on  $\widehat{G}$  agree, according to [2, Proposition F.1.4].

**Question 6.2.** *Do the two definitions of the Fell topology on  $\widehat{G}$  agree for every topological group? For every precompact group?*

## REFERENCES

- [1] L. Außenhofer, *Contributions to the Duality Theory of Abelian Topological Groups and to the Theory of Nuclear Groups*, Dissertation. Tübingen 1998; Dissertationes Mathematicae (Rozprawy Matematyczne) CCCLXXXIV, Polska Akademia Nauk, Instytut Matematyczny, Warszawa, 1999.
- [2] B. Bekka, P. de la Harpe and A. Valette, *Kazhdan's Property (T)*, Cambridge U. Press, Cambridge, 2008.
- [3] N. Bourbaki, *Espaces vectoriels topologiques*, Springer, Berlin et al., 2007.
- [4] M. J. Chasco, *Pontryagin duality for metrizable groups*, Arch. Math. **70**, 22-28 (1998).
- [5] W. Comfort, *Topological groups*, in: *Handbook of Set Theoretic Topology* (eds. K. Kunen and J. Vaughan), North-Holland, Amsterdam, 1143–1264 (1984).
- [6] W. W. Comfort, S. U. Raczkowski, F. J. Trigos-Arrieta, *The dual group of a dense subgroup*, Czechoslovak Math. Journal **54** (129), 509–533 (2004).
- [7] D. Dikranjan, D. Shakhmatov, *Quasi-convex density and determining subgroups of compact Abelian groups*, J. Math. Anal. Appl. **363**, No. 1, 42-48 (2010).
- [8] J. Dixmier, *Les  $C^*$ -algèbres et leurs représentations*, Gauthier- Villars, Paris 1969.
- [9] J. M. G. Fell, *The dual spaces of  $C^*$ -algebras*, Trans, Amer. Math. Soc. **94**, 365–403 (1960).
- [10] J. M. G. Fell, *Weak containment and induced representations of groups*, Canad. J. Math. **14**, 237–268 (1962).
- [11] M. V. Ferrer and S. Hernández, *Dual topologies on groups*. Topology and Applications, to appear.
- [12] P. de la Harpe and A. Valette, *La propriété (T) pour les groupe localement compacts*, Astérisque **175**, Soc. Math. France, 1989.
- [13] S. Hernández, S. Macario, F. J. Trigos-Arrieta, *Uncountable products of determined groups need not be determined*, J. Math. Anal. Appl. **348**, No. 2, 834-842 (2008)
- [14] K. H. Hofmann and S. A. Morris, *The Structure of Compact Groups: A Primer for Students - a Handbook for the Expert*. De Gruyter Studies in Mathematics. Berlin-New York. 2006.
- [15] D. Kazhdan, *Connection of the dual space of a group with the structure of its closed subgroups*, Funct. Annal. Appl., **1**, 63–65 (1967).
- [16] A. Lubotzky and R. J. Zimmer, *Variants of Kazhdan's Property for subgroups of semisimple groups*, Israel J. Math. **66**, 289–299 (1989).
- [17] G. Lukács, *On homomorphism spaces of metrizable groups*, J. Pure Appl. Algebra **182**, No.2-3, 263-267 (2003).
- [18] G. Margulis, *Discrete subgroups of semisimple Lie groups*, Springer-Verlag, Berlin, 1991.
- [19] W. Rudin, *Functional Analysis*, 2nd ed., McGraw-Hill, 1991.
- [20] V. Uspenskij, *Why compact groups are dyadic*, General Topology and its relations to modern analysis and algebra VI: Proc. of the 6th Prague topological Symposium 1986, Frolik Z. (ed.), Berlin: Heldermann Verlag, 1988, pp. 601-610.
- [21] V. Uspenskij, *Compact quotient spaces of topological groups and Haydon spectra*, Matem. zametki **42** (1987), No. 4, 594–602; English transl. in: Math. Notes of the Acad. Sci. USSR **42** (1987), No. 3/4, 827–831.
- [22] S. P. Wang, *On isolated points in the dual spaces of locally compact groups*, Math. Ann. **218**, 19–34 (1975).

UNIVERSITAT JAUME I, INSTITUTO DE MATEMÁTICAS DE CASTELLÓN, CAMPUS DE RIU SEC,  
12071 CASTELLÓN, SPAIN.

*E-mail address:* `mferrer@mat.uji.es`

UNIVERSITAT JAUME I, INIT AND DEPARTAMENTO DE MATEMÁTICAS, CAMPUS DE RIU  
SEC, 12071 CASTELLÓN, SPAIN.

*E-mail address:* `hernande@mat.uji.es`

DEPARTMENT OF MATHEMATICS, 321 MORTON HALL, OHIO UNIVERSITY, ATHENS, OHIO  
45701, USA

*E-mail address:* `uspenski@ohio.edu`